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HERIMITIAN SOLUTIONS TO THE EQUATION $AXA^* + BYB^* = C$, FOR HILBERT SPACE OPERATORS

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Abstract. In this paper, by using generalized inverses we have given some necessary and sufficient conditions for the existence of solutions and Hermitian solutions to some operator equations, and derived a new representation of the general solutions to these operator equations. As a consequence, we have obtained a well-known result of Dajić and Koliha.
Keywords: Hilbert space, operator equations, inner inverse, Hermitian solution.

1. Introduction and basic definitions

Let H and K be infinite complex Hilbert spaces, and $\mathbb{B}(H,K)$ the set of all bounded linear operators from H to K. Throughout this paper, the range and the null space of $A \in \mathbb{B}(H,K)$ are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ respectively. An operator $B \in \mathbb{B}(K,H)$ is said to be the inner inverse of $A \in \mathbb{B}(H,K)$ if it satisfies the equation $ABA = A$, we denote the inner inverse by A^-. An operator A is called regular if A^- exists. It is well known that $A \in \mathbb{B}(H,K)$ is regular if and only if A has closed range. There are many papers in which the basic aim is to find necessary and sufficient conditions for the existence of a solution or Hermitian solution to some matrix or operator equations using generalized inverses. In [15, 16, 18], Mitra and Navarra et al. established necessary and sufficient conditions for the existence of a common solution and gave a representation of the general common solution to the pair of matrix equations

$$A_1XB_1 = C_1 \ \ and \ \ A_2XB_2 = C_2. \tag{1.1}$$

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In [23], Wang considered the same problem for matrices over regular rings with identity. Furthermore, in [13, 16] Khatri and Mitra determined the conditions for the existence of a Hermitian solution and gave the expression of the general Hermitian solution to the matrix equation

$$AXB = C, \tag{1.2}$$

In [8] J. Groß gave the general Hermitian solution to matrix equation (1.2), where $B = A^*$.

Quaternion matrix equations and its general Hermitian solutions have attracted more attention in recent years. The reason for this is a large number of applications in control theory and many other fields, see [9, 10, 11, 12, 14, 24] and the references therein. Among them, the matrix equation

$$AXA^* + BYB^* = C, \tag{1.3}$$

has been studied by Chang and Wang in [1]. They used the generalized singular value decomposition to find necessary and sufficient conditions for the existence of real symmetric solutions. Also in [27, Corollary 3.1], Xu et al found necessary and sufficient conditions for the equation (1.3) to have a Hermitian solution.

Recently several operator equations have been extended from matrices to infinite dimensional Hilbert space, Banach space and Hilbert $\mathcal{C}^*$-modules, see [3, 4, 21], [6, 17, 22, 25, 26] and the references therein.

In this paper, our main objective is to give necessary and sufficient conditions for the existence of a Hermitian solution to the operator equation $AXA^* + BYB^* = C$. After section one where several basic definitions are assembled, in section 2, we give necessary and sufficient conditions for the existence of a common solution to the operator equations

$$A_1XB_1 = C_1 \;\; and \;\; A_2XB_2 = C_2.$$

In section 3, we apply the result of section 2 to determine new necessary and sufficient conditions for the existence of a Hermitian solution and give a representation of the general Hermitian solution to the operator equation $AXB = C$. Finally, we give some necessary and sufficient condition for the existence of a Hermitian solution to the operator equation $AXA^* + BYB^* = C$.

2. Common solutions to the operator equations $A_1XB_1 = C$ and $A_2XB_2 = C_2$

In this section, we give necessary and sufficient conditions for the existence of a common solution to the pair of equations

$$A_1XB_1 = C_1, \quad A_2XB_2 = C_2,$$

with A_1, A_2, B_1, B_2, C_1 and C_2 are linear bounded operators defined on Hilbert spaces H, K, E, L, N and G. Before enouncing our main results, we recall the following lemmas

Lemma 2.1. *[2] Let $A, B \in \mathbb{B}(H, K)$ are regular operators and $C \in \mathbb{B}(H, K)$. Then the operator equation*

$$AXB = C$$

has a solution if and only if $AA^-CB^-B = C$, or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A) \text{ and } \mathcal{R}(C^*) \subset \mathcal{R}(B^*).$$

A representation of the general solution is

$$X = A^-CB^- + U - A^-AUBB^-,$$

where $U \in \mathbb{B}(K, H)$ is an arbitrary operator.

Lemma 2.2. *[2] Let $A, B \in \mathbb{B}(H, K)$ are regular operators and $C, D \in \mathbb{B}(H, K)$. Then the pair of operators equations*

$$AX = C \quad \text{and} \quad XB = D$$

has a common solution if and only if

$$AA^-C = C, \quad DB^-B = D \quad \text{and} \quad AD = CB,$$

or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A), \quad \mathcal{R}(D^*) \subset \mathcal{R}(B^*) \quad \text{and} \quad AD = CB.$$

A representation of the general solution is

$$X = A^-C + DB^- - A^-ADB + (I_H - A^-A)V(I_H - BB^-),$$

where $V \in \mathbb{B}(H)$ is an arbitrary operator.

The following two lemmas can be deduced from a result of Patrício and Puystjens [20] originally formulated for matrix with entries in an associative ring. A simple modification shows that it applies equally well to Hilbert space operators.

Lemma 2.3. *[20] Let $A \in \mathbb{B}(H, K)$ and $B \in \mathbb{B}(E, K)$ be regular operators. Then $\begin{pmatrix} A & B \end{pmatrix} \in \mathbb{B}(H \times E, K)$ is regular if and only if $S = (I_K - AA^-)B$ is regular. In this case, the inner inverse of $\begin{pmatrix} A & B \end{pmatrix}$ is given by*

$$\begin{pmatrix} A & B \end{pmatrix}^- = \begin{pmatrix} A^- - A^-BS^-(I_K - AA^-) \\ S^-(I_K - AA^-) \end{pmatrix}.$$

Lemma 2.4. *[3] Let $A \in \mathbb{B}(H, K)$ and $B \in \mathbb{B}(H, E)$ be regular operators. Then the regularity of any one of the following operators implies the regularity of the remaining three operators*

$$D=B(I_H-A^-A),\ M=A(I_H-B^-B),\ \begin{pmatrix} A \\ B \end{pmatrix} \quad and \quad \begin{pmatrix} B \\ A \end{pmatrix}.$$

In this case, the inner inverse of $\begin{pmatrix} A \\ B \end{pmatrix}$ *is given by*

$$\begin{pmatrix} A \\ B \end{pmatrix}^- = \begin{pmatrix} (I_H-B^-B)M^- & B^- -(I_H-B^-B)M^-AB^- \end{pmatrix}.$$

Lemma 2.5. *[2] Suppose that* $A_1 \in \mathbb{B}(H,K)$, $A_2 \in \mathbb{B}(H,E)$, $B_1 \in \mathbb{B}(L,G)$, $B_2 \in \mathbb{B}(N,G)$, $S_1 = A_2(I_H - A_1^- A_1)$ *and* $M_1 = (I_G - B_1B_1^-)B_2$ *are regular operators. Then*

$$T_1 = (I_E - S_1S_1^-)A_2A_1^- \ \ and \ \ D_1 = B_1^- B_2(I_N - M_1^- M_1),$$

are regular with inner inverses $T_1^- = A_1A_2^-$ *and* $D_1^- = B_2^- B_1$.

In the following theorem, we give necessary and sufficient conditions for the existence of a common solution of the operator equations

$$A_1XB_1 = C_1, \quad A_2XB_2 = C_2$$

Theorem 2.1. *Suppose that* $A_1 \in \mathbb{B}(H,K)$, $A_2 \in \mathbb{B}(H,E)$, $B_1 \in \mathbb{B}(L,G)$, $B_2 \in \mathbb{B}(N,G)$, $M_1 = (I_G - B_1B_1^-)B_2$ *and* $S_1 = A_2(I_H - A_1^- A_1)$ *are regular operators and* $C_1 \in \mathbb{B}(L,K)$, $C_2 \in \mathbb{B}(N,E)$. *Then the following statements are equivalent*

1. *The pair of equations (1.1) have a common solution* $X \in \mathbb{B}(G,H)$.
2. *There exists two operators* $U \in \mathbb{B}(N,K)$ *and* $V \in \mathbb{B}(L,E)$, *such that the operator equation* $AXB = C$ *is solvable, where*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}.$$

3. *For* $i = 1,2$, $\mathcal{R}(C_i) \subset \mathcal{R}(A_i)$, $\mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*)$ *and*

$$T_1C_1D_1 = T_2C_2D_2,$$

where $T_1 = (I_E - S_1S_1^-)A_2A_1^-$, $T_2 = (I_E - S_1S_1^-)$, $D_1 = B_1^- B_2(I_N - M_1^- M_1)$ *and* $D_2 = (I_N - M_1^- M_1)$.

Proof.

(1) $\Leftrightarrow$ (2) The equivalence is easily established.

(2) $\Rightarrow$ (3) According to Lemma 2.1, the operator equation $AXB = C$ has a solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \quad and \quad \mathcal{R}(C^*) \subset \mathcal{R}(B^*),$$

then, we deduce that

$$\text{(2.1)} \qquad for \quad i = 1, 2, \quad \mathcal{R}(C_i) \subset \mathcal{R}(A_i) \quad and \quad \mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*).$$

On the other hand, we have

$$T_1C_1D_1 = (I_E - S_1S_1^-)A_2A_1^-C_1B_1^-B_2(I_N - M_1^-M_1)$$

$$\text{(2.2)} \qquad = (I_E - S_1S_1^-)A_2A_1^-A_1X_0B_1B_1^-B_2(I_N - M_1^-M_1),$$

where X_0 is the common solution to the pair of equations (1.1).
Let

$$S_1 = A_2(I_H - A_1^-A_1) \quad and \quad M_1 = (I_G - B_1B_1^-)B_2.$$

This implies that

$$\text{(2.3)} \qquad A_2A_1^-A_1 = A_2 - S_1 \quad and \quad B_1B_1^-B_2 = B_2 - M_1.$$

We insert (2.3) in (2.2) to obtain

$$\text{(2.4)} \qquad T_1C_1D_1 = T_2C_2D_2.$$

From (2.1) and (2.4), we deduce that (2) $\Rightarrow$ (3).

Conversely, since

$$T_1C_1D_1 = T_2C_2D_2.$$

Then

$$\mathcal{R}(T_2C_2) \subset \mathcal{R}(T_1) \quad and \quad \mathcal{R}(D_1^*C_1^*) \subset \mathcal{R}(D_2^*).$$

By applying Lemma 2.2, there exist $U \in \mathbb{B}(N, K)$ which is the common solution to the pair of equations

$$\text{(2.5)} \qquad \begin{cases} T_1U = T_2C_2 \\ UD_2 = C_1D_1, \end{cases}$$

given by

$$\text{(2.6)} \qquad U = C_1D_1 + T_1^-(I_E - S_1S_1^-)C_2M_1^-M_1 + (A_1A_1^- - T_1^-T_1)ZM_1^-M_1,$$

where $Z \in \mathbb{B}(N, K)$ is an arbitrary operator.
On other hand, since

$$T_1C_1D_1 = T_2C_2D_2.$$

Then

$$\mathcal{R}(T_1C_1) \subset \mathcal{R}(T_2) \quad and \quad \mathcal{R}(D_2^*C_2^*) \subset \mathcal{R}(D_1^*).$$

It follows from Lemma 2.2 that there exist $V \in \mathbb{B}(L, E)$ which is the common solution to the pair of equations

$$\text{(2.7)} \qquad \begin{cases} T_2V = T_1C_1 \\ VD_1 = C_2D_2, \end{cases}$$

given by

$$V = T_1C_1 + S_1S_1^- C_2(I_N - M_1^- M_1)D_1^- + S_1S_1^- Z'(B_1^- B_1 - D_1D_1^-), \tag{2.8}$$

where $Z' \in \mathbb{B}(L, E)$ is an arbitrary operator.
Thus, there exists $U \in \mathbb{B}(N, K)$ and $V \in \mathbb{B}(L, E)$ solutions to the pair of equations (2.5), (2.7) and as for $i = 1, 2$, we have $A_iA_i^- C_i = C_i$ and $C_iB_i^- B_i = C_i$, we obtain

$$\begin{aligned} &AA^- CB^- B = \\ &= \begin{pmatrix} A_1A_1^- C_1B_1^- B_1 & A_1A_1^-(C_1D_1 + UM_1^- M_1) \\ (T_1C_1 + S_1S_1^- V)B_1^- B_1 & T_1(C_1D_1 + UM_1^- M_1) + S_1S_1^-(V D_1 + C_2M_1M_1^- \end{pmatrix} \\ &= C. \end{aligned}$$

So that, the operator equation $AXB = C$ is solvable and (3) $\Rightarrow$ (2). $\square$

Theorem 2.2. *Suppose that $A_1 \in \mathbb{B}(H, K)$, $A_2 \in \mathbb{B}(H, E)$, $B_1 \in \mathbb{B}(L, G)$, $B_2 \in \mathbb{B}(N, G)$, $M_1 = (I_G - B_1B_1^-)B_2$ and $S_1 = A_2(I_H - A_1^- A_1)$ are regular operators and $C_1 \in \mathbb{B}(L, K)$, $C_2 \in \mathbb{B}(N, E)$, when any one of the conditions (2), (3) of Theorem 2.1 holds, a general common solution to the pair of equations (1.1) is given by*

$$\begin{aligned} X \;=\;& (A_1^- C_1 + (I_H - A_1^- A_1)S_1^-(V - A_2A_1^- C_1))B_1^-(I_G - B_2M_1^-(I_G - B_1B_1^-)) \\ +\;& (A_1^- U + (I_H - A_1^- A_1)S_1^-(C_2 - A_2A_1^- U))M_1^-(I_G - B_1B_1^-) + F \\ -\;& (A_1^- A_1 + (I_H - A_1^- A_1)S_1^- S_1)F(B_1B_1^- + M_1M_1^-(I_G - B_1B_1^-)), \end{aligned} \tag{2.9}$$

where $F \in \mathbb{B}(G, H)$ is an arbitrary operator and U, V are given by

$$\begin{cases} U = C_1B_1^- B_2(I_N - M_1^- M_1) + A_1A_2^-(I_E - S_1S_1^-)C_2M_1^- M_1 + A_1A_1^- ZM_1^- M_1 \\ \quad -A_1A_2^-(I_E - S_1S_1^-)A_2A_1^- ZM_1^- M_1, \\ \text{and} \\ V = (I_E - S_1S_1^-)A_2A_1^- C_1 + S_1S_1^- C_2(I_N - M_1^- M_1)B_2^- B_1 + S_1S_1^- Z^{'} B_1^- B_1 \\ \quad -S_1S_1^- Z^{'} B_1^- B_2(I_N - M_1^- M_1)B_2^- B_1, \end{cases}$$

where $Z \in \mathbb{B}(N, K)$ and $Z' \in \mathbb{B}(L, E)$ are arbitrary operators.

Proof. From Theorem 2.1, we get that the pair of equations (1.1) has a common solution equivalently the two conditions (2) and (3) holds.
On the other hand, since the pair of equations (1.1) is equivalent to

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}. \tag{2.10}$$

According to Lemma 2.3 and Lemma 2.4, we have

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in \mathbb{B}(H, K \times E) \quad \text{and} \quad \begin{pmatrix} B_1 & B_2 \end{pmatrix} \in \mathbb{B}(L \times N, G)$$

are regular with inner inverses

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- = \begin{pmatrix} (I_E - A_2^- A_2)S_1^- & A_2^- - (I_E - A_2^- A_2)S_1^- A_1 A_2^- \end{pmatrix}, \tag{2.11}$$

and

$$\begin{pmatrix} B_1 & B_2 \end{pmatrix}^- = \begin{pmatrix} B_1^- - B_1^- B_2 M_1^- (I_G - B_1 B_1^-) \\ M_1^- (I_G - B_1 B_1^-) \end{pmatrix}, \tag{2.12}$$

respectively.
Using Lemma 2.1, we deduce that the general solution of (2.10) is given by

$$\begin{aligned} X \;=\; & \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^- + \\ & + F - \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} F \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^- . \end{aligned} \tag{2.13}$$

By substituting (2.11) and (2.12) in (2.13), we get the solution X as defined in (2.9) such that U, V are given in (2.6) and (2.8) respectively and $F \in \mathbb{B}(G, H)$ is an arbitrary operator. □

3. Hermitian solutions to the operator equations $AXB = C$ and $AXA^* + BYB^* = C$

Based on Theorem 2.1 and Theorem 2.2, in this section we give necessary and sufficient conditions for the existence of Hermitian solutions to the operator equations

$$AXB = C \quad \textit{and} \quad AXA^* + BYB^* = C$$

and obtain the general Hermitian solution to those operator equations respectively. Before enouncing our main results we have the following lemma

Lemma 3.1. *Let $A \in \mathbb{B}(H, K)$ and $B \in \mathbb{B}(K, H)$, such that A, B, $S_1 = B^*(I_H - A^- A)$ and $M_1 = (I_H - BB^-)A^*$ are regular. Then the operator equation*

$$AXB = C,$$

has a Hermitian solution if and only if the pair of operator equations

$$AXB = C \quad \textit{and} \quad B^* X A^* = C^* \tag{3.1}$$

has a common solution, a representation of the general Hermitian solution to $AXB = C$ is of the form

$$X_H = \frac{X + X^*}{2},$$

where X is the representation of the general common solution to the equations (3.1).

Proof. From Theorem 2.1 the pair of operator equations (3.1) has a common solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \ \text{and} \ \mathcal{R}(C^*) \subset \mathcal{R}(B^*),$$

and

$$(I_K - S_1S_1^-)B^*A^-CB^-A^*(I_K - M_1^-M_1) = (I_K - S_1S_1^-)C^*(I_K - M_1^-M_1).$$

A representation of the general common solution to equations (3.1) is given by (2.9) in Theorem 2.2, where $A_1 = A$, $B_1 = B$, $C_1 = C$, $A_2 = B^*$, $B_2 = A^*$ and $C_2 = C^*$. Clearly X_H is a Hermitian solution to (1.2). $\square$

From the above proof and Theorem 2.2, we obtain the following corollary.

Corollary 3.1. *Let $A \in \mathbb{B}(H,K)$, $B \in \mathbb{B}(K,H)$, $M_1 = (I_H - BB^-)A^*$ and $S_1 = B^*(I_H - A^-A)$ are regular operators and $C \in \mathbb{B}(K)$. Then the operator equation*

$$AXB = C,$$

has a Hermitian solution if and only if

1. $\mathcal{R}(C) \subset \mathcal{R}(A)$ *and* $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$
2. $(I_K - S_1S_1^-)B^*A^-CB^-A^*(I_K - M_1^-M_1) = (I_K - S_1S_1^-)C^*(I_K - M_1^-M_1)$.

In this case, a representation of the general Hermitian solution is of the form

$$X_H = \frac{X + X^*}{2},$$

where

$$\begin{aligned} X &= (A^-C + (I_H - A^-A)S_1^-(V - B^*A^-C))B^-(I_H - A^*M_1^-(I_H - BB^-)) \\ &\quad +(A^-U + (I_H - A^-A)S_1^-(C^* - B^*A^-U))M_1^-(I_H - BB^-) + F \\ &\quad - (A^-A + (I_H - A^-A)S_1^-S_1)F(BB^- + M_1M_1^-(I_H - BB^-), \end{aligned} \tag{3.2}$$

where $F \in \mathbb{B}(H)$ is an arbitrary operator and U, V are given by

$$\begin{cases} U = CB^-A^*(I_K - M_1^-M_1) + A(B^*)^-(I_K - S_1S_1^-)C^*M_1^-M_1 + AA^-ZM_1^-M_1 \\ \quad -A(B^*)^-(I_K - S_1S_1^-)B^*A^-ZM_1^-M_1 \\ \text{and} \\ V = (I_K - S_1S_1^-)B^*A^-C + S_1S_1^-C^*(I_K - M_1^-M_1)(A^*)^-B + S_1S_1^-Z'B^-B \\ \quad -S_1S_1^-Z'B^-A^*(I_K - M_1^-M_1)(A^*)^-B, \end{cases}$$

where $Z, Z' \in \mathbb{B}(K)$ are arbitrary operators.

Corollary 3.2. *Let $A \in \mathbb{B}(H,K)$, $C \in \mathbb{B}(K)$ such that A is regular and $C^* = C$. Then the operator equation*

$$AXA^* = C$$

has a Hermitian solution $X \in \mathbb{B}(H)$ if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A)$$

In this case, a representation of the general Hermitian solution is

$$X_H = A^- C(A^-)^* + F - A^- AF(A^- A)^*, \tag{3.3}$$

where $F \in \mathbb{B}(H)$ is an arbitrary Hermitian operator.

Proof. We put $B = A^*$ in Corollary 3.1 we get the result. □

As a consequence of Corollary 3.1 we obtain the well-known Theorem of Alegra Dajić and J.J. Koliha [3, Theorem 3.1].

Corollary 3.3. *[3, Theorem 3.1] Let $A, C \in \mathbb{B}(H,K)$ such that A is a regular operator. Then the operator equation*

$$AX = C$$

has a Hermitian solution $X \in \mathbb{B}(H)$ if and only if

$$AA^- C = C \quad \text{and} \quad AC^* \text{ is Hermitian.}$$

The general Hermitian solution is of the form

$$X_H = A^- C + (I_H - A^- A)(A^- C)^* + (I_H - A^- A)Z'(I_H - A^- A)^*,$$

where $Z' \in \mathbb{B}(H)$ is an arbitrary Hermitian operator.

Proof. By applying Corollary 3.1, the operator equation $AX = C$ has a Hermitian solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A),$$

which is equivalent to

$$AA^- C = C,$$

and

$$(I_H - I_H + A^- A)A^- CA^* = (I_H - I_H + A^- A)C^*,$$

this implies that

$$CA^* = AC^*.$$

Hence, AC^* is Hermitian. In this case,

$$\begin{aligned} X &= [A^- C + (I_H - A^- A)(A^- C + (I_H - A^- A)C^*(A^*)^- + \\ &+ (I_H - A^- A)Z'(I_H - A^- A)^* - A^- C)], \\ &= A^- C + (I_H - A^- A)(A^- C)^* + (I_H - A^- A)Z'(I_H - A^- A)^*. \end{aligned}$$

It follows that,

$$\begin{aligned} X_H &= \frac{X+X^*}{2}, \\ &= A^-C + (I_H - A^-A)(A^-C)^* + (I_H - A^-A)Z'(I_H - A^-A)^*. \end{aligned}$$

□

Theorem 3.1. *Let $A, B \in \mathbb{B}(H,K)$ and $A_1 = (I_K - AA^-)B$, $C_1 = (I_K - AA^-)C$ and $S_2 = B(I_H - A_1^- A_1)$ be all regular and $C \in \mathbb{B}(K)$ is Hermitian. Then the operator equation*

$$AXA^* + BYB^* = C,$$

has a Hermitian solution if and only if

1. $A_1 A_1^-(I_K - AA^-)C(B^*)^- B^* = (I_K - AA^-)C$
2. $(I_K - S_2 S_2^-)[C - BA_1^-(I_K - AA^-)C(B^*)^- B^*](I_K - (A^-)^* A^*) = 0.$

In this case, a representation of the general Hermitian solution is of the form

$$(X_H, Y_H) = \left(\frac{X+X^*}{2}, \frac{Y+Y^*}{2}\right),$$

where X and Y are given by

$$\begin{cases} X = A^-(C - BYB^*)(A^*)^- + F - A^-AF(A^-A)^* \\ \text{and} \\ Y = A_1^-(I_K - AA^-)C(B^*)^- + \\ \quad +(I_H - A_1^-A_1)S_2^-[V - BA_1^-(I_K - AA^-)C](B^*)^- + U \\ \quad -[A_1^-A_1 + (I_H - A_1^-A_1)S_2^-S_2]UB^*(B^*)^-, \end{cases}$$

and

$$\begin{aligned} V &= (I_K - S_2S_2^-)BA_1^-(I_K - AA^-)C + S_2S_2^-C(I_K - (A^-)^*A^*)(A_1^*)^- B^* \\ &+ S_2S_2^-Z(B^*)^-(I_H - A_1^*(A_1^-)^*)B^*, \end{aligned}$$

with $F \in \mathbb{B}(H)$, $U \in \mathbb{B}(H)$ and $Z \in \mathbb{B}(K)$ are arbitrary Hermitian operators.

Proof. The operator equation (1.3) is equivalent to

$$AXA^* = C - BYB^*. \tag{3.4}$$

Applying Corollary 3.2, the operator equation (3.4) has a Hermitian solution if and only if

$$\begin{aligned} \mathcal{R}(C - BYB^*) \subset \mathcal{R}(A) &\Leftrightarrow AA^-(C - BYB^*) = (C - BYB^*), \\ &\Leftrightarrow (I - AA^-)(C - BYB^*) = 0. \end{aligned} \tag{3.5}$$

Then, (3.5) is equivalent to the operator equation

$$A_1YB^* = C_1, \tag{3.6}$$

with $A_1 = (I_K - AA^-)B$, and $C_1 = (I_K - AA^-)C$.
From Corollary 3.1, the operator equation (3.6) has a Hermitian solution if and only if

$$\begin{aligned} \mathcal{R}(C_1) \subset \mathcal{R}(A_1) \quad &\Leftrightarrow \quad A_1A_1^-C_1 = C_1, \\ &\Leftrightarrow \quad A_1A_1^-(I_K - AA^-)C = (I_K - AA^-)C, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \mathcal{R}(C_1^*) \subset \mathcal{R}(B) \quad &\Leftrightarrow \quad C_1(B^*)^-B^* = C_1, \\ &\Leftrightarrow \quad (I_K - AA^-)C(B^*)^-B^* = (I_K - AA^-)C. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get

$$A_1A_1^-(I_K - AA^-)C(B^*)^-B^* = (I_K - AA^-)C.$$

On the other hand, we have

$$(I_K - S_2S_2^-)BA_1^-(I_K - AA^-)C(B^*)^-A_1^* = (I_K - S_2S_{2-})C(I_K - (A^-)^*A^*).$$

This implies that

$$(I_K - S_2S_2^-)[C - BA_1^-(I_K - AA^-)C(B^*)^-B^*](I_K - (A^-)^*A^*) = 0.$$

A representation of the general Hermitian solution to the operator equation (3.6) is of the form

$$Y_H = \frac{Y + Y^*}{2},$$

where Y is given by (3.2) in Corollary 3.1 such that $A = A_1$, $B = B^*$ and $C = C_1$

$$\begin{aligned} Y \;=\;& A_1^-(I_K - AA^-)C(B^*)^- + (I_H - A_1^-A_1)S_2^-[V - BA_1^-(I_K - AA^-)C](B^*)^- + \\ +\;& U - [A_1^-A_1 + (I_H - A_1^-A_1)S_2^-S_2]UB^*(B^*)^-, \end{aligned}$$

and

$$\begin{aligned} V \;=\;& (I_K - S_2S_2^-)BA_1^-(I_K - AA^-)C + S_2S_2^-C(I_K - (A^-)^*A^*)(A_1^*)^-B^* + \\ &+ S_2S_2^-Z(B^*)^-(I_H - A_1^*(A_1^-)^*)B^*, \end{aligned}$$

with $U \in \mathbb{B}(H)$ and $Z \in \mathbb{B}(K)$ are arbitrary Hermitian operators.
We return to the operator equation

$$AXA^* = C - BYB^*,$$

in order to find the Hermitian solution X.

By Corollary 3.2, the operator equation (3.4) has a Hermitian solution if and only if

$$\mathcal{R}(C - BYB^*) \subset \mathcal{R}(A).$$

So the operator equation (3.4) has a Hermitian solution X_H given by

$$X_H = A^-(C - BYB^*)(A^*)^- + F - A^-AF(A^-A)^*,$$

with $F \in \mathbb{B}(H)$ is an arbitrary Hermitian operator. $\square$

4. Conclusions

This paper gives necessary and sufficient conditions for the existence of a common solution to the pair of equations

$$A_1XB_1 = C_1 \ \ \textit{and} \ \ A_2XB_2 = C_2;$$

We have applied this result to determine new necessary and sufficient conditions for the existence of Hermitian solution and given a representation of the general Hermitian solution to the operator equation

$$AXB = C.$$

Then, we have given necessary and sufficient conditions for the existence of Hermitian solution and a representation of the general Hermitian solution to the operator equation

$$AXA^* + BYB^* = C.$$

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